Compact Nodal Analysis With Controlled Sources Modeled by Ideal Operational Amplifiers

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***Abstract*—This paper describes the formulation of a compact form of nodal analysis, obtained by the use of ideal operational amplifiers in the modeling of voltage sources and current-controlled sources. The method produces a system of equations that is never larger than a simple nodal system, at the expense of a simple preprocessing. The method is particularly suitable for sensitivity analysis programs, because only the circuit variables required for sensitivity analysis are computed.**

I. Introduction

A rather long introduction is necessary to situate the subject of the paper and the notation used.

The Nodal Analysis Method

The most commonly used method for circuit analysis is the "nodal analysis" method, that consists in writing for all the n nodes in a circuit, with the exception of a "reference" node, equations in the form:

$$\Sigma \text{ branch currents leaving the node} = \\ = \Sigma \text{ current sources feeding current to the node}$$

The branch currents are expressed as functions of the voltages between the nodes and the reference node (nodal voltages). For a linear resistive circuit, a linear system of equations results, in the form $\mathbf{G}_n\,\mathbf{e} = \mathbf{I}_s$, where $\mathbf{G}_n$ is the $n \times n$ nodal conductance matrix, $\mathbf{e}$ is the nodal voltages vector, and $\mathbf{I}_s$ is the current sources vector.

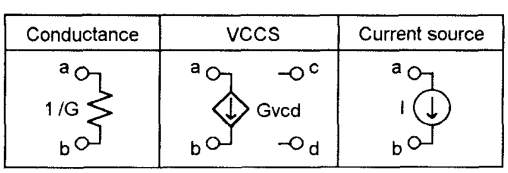

Fig. 1. Circuit elements allowed in the basic nodal analysis method.

The analysis of linear time-invariant circuits in the sinusoidal steady state or with Laplace transforms is structurally identical. The nodal analysis of nonlinear and/or time-variant circuits can be done by methods the have as fundamental steps the solving of linear resistive circuits [1][5]. The discussions that follow use resistive linear circuits as examples, but are valid for any of these extensions.

The basic nodal analysis method accepts as elements (fig. 1) only conductances, current sources, and voltage-controlled current sources (VCCS). The nodal system of equations can be constructed in a systematic way in a computer program by the algorithm [1]:

1. Fill with zeros $\mathbf{G}_n$ and $\mathbf{I}_s$.
2. For all the circuit elements, add the corresponding "stamps" (table 1) to the system.

The columns of the entries in $\mathbf{G}_n$ correspond to the positions of the unknowns. The positions with dots are just for reference.

TABLE 1. STAMPS FOR THE ELEMENTS IN FIG. 1.

Conductance	$\begin{matrix} a \\ b \end{matrix}\begin{bmatrix} +G & -G \\ -G & +G \end{bmatrix}\begin{bmatrix} e_a \\ e_b \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$
VCCS	$\begin{matrix} a \\ b \\ c \\ d \end{matrix}\begin{bmatrix} \cdot & \cdot & +G & -G \\ \cdot & \cdot & -G & +G \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} e_a \\ e_b \\ e_c \\ e_d \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$
Current source	$\begin{matrix} a \\ b \end{matrix}\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} e_a \\ e_b \end{bmatrix} = \begin{bmatrix} -I \\ +I \end{bmatrix}$

The Modified Nodal Analysis Method (MNA)

The MNA method [1][3][5] is similar to the nodal analysis method, but includes currents in voltage-controlled branches as new unknowns, and the corresponding branch equations as new equations. These changes allow the inclusion of ideal voltage sources and the other three types of controlled sources, elements without a direct nodal representation, at the expense of a larger system of equations.

Coupling special elements through gyrators

The MNA method is equivalent to a normal nodal method where the special elements are coupled to the circuit by gyrators. This idea is not new. It is suggested in [1] (problem 4-19), and also discussed in [6]. Fig. 2 shows equivalent circuits with nodal representation that are equivalent to four basic special elements. In all cases, the branches of the special elements that contain voltage sources or short-circuits are converted into their duals, and connected to the network through pairs of VCCSs, that correspond to unitary gyrators, as shown. The nodal voltages

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in the extra nodes (x and y) are numerically identical to the currents in the original voltage sources or short-circuits, the extra unknowns introduced by MNA. The stamps of the equivalents in fig. 2 in the nodal system are shown in table 2. They are exactly the same used in the MNA method.

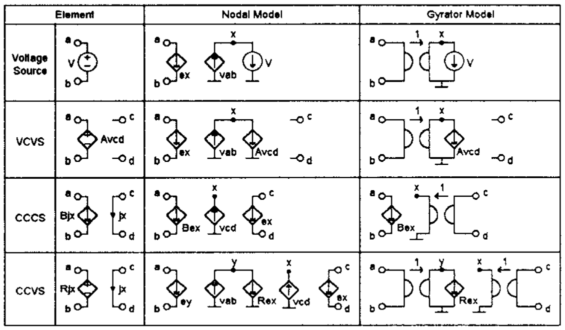

Fig. 2. Circuit elements that cannot be included directly in normal nodal analysis, and their equivalent "nodal models", that transform exactly the nodal analysis in a "modified" nodal analysis.

TABLE 2. STAMPS FOR THE ELEMENTS IN FIG. 2.

Voltage source	$\begin{matrix}a\\b\\c\end{matrix}\begin{bmatrix}\cdot & \cdot & +1\\ \cdot & \cdot & -1\\ -1 & +1 & \cdot\end{bmatrix}\begin{bmatrix}e_a\\e_b\\e_x\end{bmatrix}=\begin{bmatrix}\cdot\\ \cdot\\ -V\end{bmatrix}$
VCVS	$\begin{matrix}a\\b\\c\\d\\x\end{matrix}\begin{bmatrix}\cdot & \cdot & \cdot & \cdot & +1\\ \cdot & \cdot & \cdot & \cdot & -1\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ -1 & +1 & +A & -A & \cdot\end{bmatrix}\begin{bmatrix}e_a\\e_b\\e_c\\e_d\\e_x\end{bmatrix}=\begin{bmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}$
CCCS	$\begin{matrix}a\\b\\c\\d\\x\end{matrix}\begin{bmatrix}\cdot & \cdot & \cdot & \cdot & +B\\ \cdot & \cdot & \cdot & \cdot & -B\\ \cdot & \cdot & \cdot & \cdot & +1\\ \cdot & \cdot & \cdot & \cdot & -1\\ \cdot & \cdot & -1 & +1 & \cdot\end{bmatrix}\begin{bmatrix}e_a\\e_b\\e_c\\e_d\\e_x\end{bmatrix}=\begin{bmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}$
CCVS	$\begin{matrix}a\\b\\c\\d\\x\\y\end{matrix}\begin{bmatrix}\cdot & \cdot & \cdot & \cdot & \cdot & +1\\ \cdot & \cdot & \cdot & \cdot & \cdot & -1\\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot\\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot\\ \cdot & \cdot & -1 & +1 & \cdot & \cdot\\ -1 & +1 & \cdot & \cdot & +R & \cdot\end{bmatrix}\begin{bmatrix}e_a\\e_b\\e_c\\e_d\\e_x\\e_y\end{bmatrix}=\begin{bmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix}$

The Ideal Operational Amplifier

In the MNA method, an ideal operational amplifier (fig. 3) is included by adding as unknown its output current, and including the equation $e_c=e_d$. The corresponding nodal model reduces to two VCCSs, one with the output voltage controlling the other.

The condition $v_{cd}=0$ is forced by the input VCCS because its output current must be zero. As e_x controls the output current, the circuit is solvable only if there is some external feedback connection from the output to the input of the op. amp. The model corresponds to an ideal infinite-gain voltage amplifier, or nullator-norator pair. The nodal voltage e_x is numerically equal to the current through the op. amp. output. By this model, the ideal op. amp. stamp, the same of the MNA method, is:

$$\begin{matrix}a\\b\\c\\d\\x\end{matrix}\begin{bmatrix}\cdot & \cdot & \cdot & \cdot & +1\\ \cdot & \cdot & \cdot & \cdot & -1\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & -1 & +1 & \cdot\end{bmatrix}\begin{bmatrix}e_a\\e_b\\e_c\\e_d\\e_x\end{bmatrix}=\begin{bmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix} \quad (1)$$

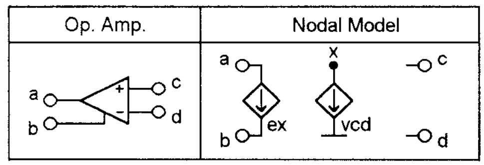

Fig. 3. Ideal operational amplifier, and its nodal model.

Ideal op. amps. can be modeled in a more efficient way [4]: In an ideal operational amplifier, the input node voltages are equal. To reduce them to a single unknown corresponds to add the c and d columns of $\mathbf{G}_n$ (or to remove one column if one input is grounded). The output "current" unknown e_x can be eliminated if the two equations corresponding to the output nodes a and b are added (or one output node equation eliminated if the other output terminal is grounded). With these reductions, the ideal op. amp. removes one equation of the system. The simplified "stamp" for the ideal op. amp. can be represented as:

$$\begin{matrix}\left\{\begin{matrix}a\\b\end{matrix}\right.\\c\\d\end{matrix}\begin{bmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{bmatrix}\begin{bmatrix}e_a\\e_b\\ \left\{\begin{matrix}e_c\\e_d\end{matrix}\right.\end{bmatrix}=\begin{bmatrix}\cdot\\ \cdot\\ \cdot\\ \cdot\end{bmatrix} \quad (2)$$

The brackets mean that, after all the elements' stamps are in place, the indicated equations and columns (corresponding to the unknowns turned equal) shall be added. If one of the indicated equations or unknowns refers to the reference node, the other shall be eliminated, what is equivalent to perform the additions in the indefinite admittance matrix and its excitation.

This idea suggested the models for the special elements described in the next section, that are similar to the models in fig. 2, but use ideal operational amplifiers instead of some VCCSs, with the objective of obtaining a final system that is much more compact.

The resulting system is somewhat similar to the one obtained with the also compact "two-graph modified nodal" formulation [7].

II. Economical Nodal Models For Special Elements Using Ideal Operational Amplifiers

Simpler models for the special elements can be obtained by the elimination of the voltage source "current" unknowns and nodal voltages in one side of real of virtual short-circuits. The models shown in fig. 4 cause these eliminations, if the op. amps. are treated in the simplified way. All the models retain the order of the system of equations, with the added variables removed by the op. amps. The current variables are not computed, with the exception of the input currents of the current-controlled sources. A set of stamps for the special elements is shown in table 3, where the unknowns and equations eliminated by the op. amps. with grounded input or output are directly omitted. The brackets indicate the sums that are to be made when the stamps of all the elements are in place, as in the case of the simple op. amp. Note that some stamps add a new equation or a new unknown, but never both.

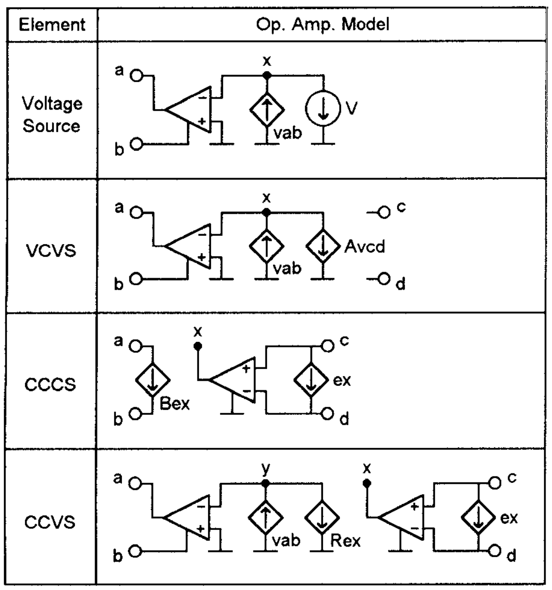

Fig. 4. Models for the special elements using ideal op.amps.

It is interesting to examine how the models in fig. 4 relate to the MNA equivalent models in fig. 2. In the case of the voltage sources, if the transconductance of the output VCCS is increased to infinity, the model remains functional, but e_x (e_y for the CCVS) reduces to 0, and the output VCCS is transformed into an ideal operational amplifier. In the current-controlled sources, the VCCS connected to node x, in open circuit, behaving as an infinite-gain voltage amplifier, and so is directly equivalent to an ideal operational amplifier.

A simpler model for the CCVS can be obtained if $R\neq 0$. It is similar to the input circuits used for the current-controlled sources in fig. 4, and is shown in fig. 5. Its equivalent stamp is, with $g=1/R$:

$$\begin{matrix} a \\ b \\ c \\ d \end{matrix}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ +g & -g & \cdot & \cdot \\ -g & +g & \cdot & \cdot \end{bmatrix}\begin{bmatrix} e_a \\ e_b \\ e_c \\ e_d \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad (3)$$

The use of this model decreases by one the size of the nodal system. It can also be used as a "current meter", with R=1 and node "b" grounded, measuring the current in a short circuit between the nodes "c" and "d" as the voltage at node "a".

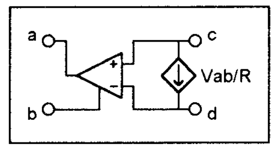

Fig. 5. Alternative simpler model for the CCVS, when $R\neq 0$.

TABLE 3. STAMPS FOR THE ELEMENTS IN FIG. 4.

Voltage source	$\begin{matrix} a \\ b \\ x \end{matrix}\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ -1 & +1 \end{bmatrix}\begin{bmatrix} e_a \\ e_b \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ -V \end{bmatrix}$
VCVS	$\begin{matrix} a \\ b \\ c \\ d \\ x \end{matrix}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & +1 & +A & -A \end{bmatrix}\begin{bmatrix} e_a \\ e_b \\ e_c \\ e_d \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$
CCCS	$\begin{matrix} a \\ b \\ c \\ d \end{matrix}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & +B \\ \cdot & \cdot & \cdot & \cdot & -B \\ \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix}\begin{bmatrix} e_a \\ e_b \\ e_c \\ e_d \\ e_x \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$
CCVS	$\begin{matrix} a \\ b \\ c \\ d \\ x \end{matrix}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & +1 & \cdot & \cdot & +R \end{bmatrix}\begin{bmatrix} e_a \\ e_b \\ e_c \\ e_d \\ e_x \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$

The simplifications above can be done by a simple preprocessing, that generates two sets of pointers that indicate where the nodal equations ($\mathbf{G}_n$ and $\mathbf{I}_s$ lines) and unknowns ($\mathbf{G}_n$ columns) come to be in the final reduced system. The stamps of all the elements are then added as indicated by the pointers, and the results taken where indicated by the pointers for unknowns after the solution of the system.

III. Example

An example circuit containing all the special elements is shown in fig. 6. The circuit has 5 nodes, but the construction of the nodal system using the stamps in fig. 2 (MNA) adds 5 currents ($j_6,\ldots,j_{10}$) as unknowns, increasing the order to 10.

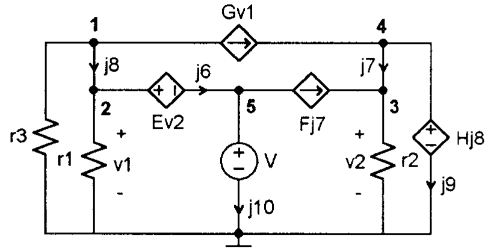

Fig. 6. Example circuit.

The resulting system would be:

$$\left[\begin{array}{ccccc:ccccc} g_3+G & & & & & & & 1 & & \\ & g_1 & & & & 1 & & -1 & & \\ & & g_2 & & & & -1-F & & & \\ -G & & & & & & 1 & & 1 & \\ & & & & & -1 & F & & & 1 \\ \hdashline & -1 & E & & 1 & & & & & \\ & & 1 & -1 & & & & & & \\ -1 & 1 & & & & & & & & \\ & & & -1 & & & & H & & \\ & & & & -1 & & & & & \end{array}\right]\left[\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ \hdashline j_6 \\ j_7 \\ j_8 \\ j_9 \\ j_{10} \end{array}\right]=\left[\begin{array}{c} \\ \\ \\ \\ \\ \hdashline \\ \\ \\ \\ -V \end{array}\right] \tag{4}$$

The models in fig. 4 produce a system with only 5 equations. Before the equation and column sums, the system is:

$$\begin{array}{r} e_1 \\ \leftarrow e_2 \\ e_3 \\ \leftarrow e_4 \\ \leftarrow e_5 \\ j_6 \\ j_9 \\ j_{10} \end{array}\left[\begin{array}{ccccc:cc} g_3+G & & & & & & 1 \\ & g_1 & & & & & -1 \\ & & g_2 & & & -1-F & \\ -G & & & & & 1 & \\ & & & & & F & \\ \hdashline & -1 & E & & 1 & & \\ & & & -1 & & & H \\ & & & & -1 & & \end{array}\right]\left[\begin{array}{c} \left\{\begin{array}{c} e_1 \\ e_2 \end{array}\right. \\ \left\{\begin{array}{c} e_3 \\ e_4 \end{array}\right. \\ e_5 \\ \hdashline j_7 \\ j_8 \end{array}\right]=\left[\begin{array}{c} \\ \\ \\ \\ \\ \hdashline \\ \\ -V \end{array}\right] \tag{5}$$

The VCVS adds the equations e_2 and e_5, and so the grounded voltage sources cause the elimination of equations e_2, e_4, and e_5. The columns corresponding to terminals of short-circuits are also added, with their terminal nodal voltages reduced to single unknowns ($\{e_1, e_2\}$, $\{e_3, e_4\}$). The final system is (6). All the nodal voltages are computed, and also the currents in short-circuits.

$$\begin{array}{r} e_1 \\ e_3 \\ j_6 \\ j_9 \\ j_{10} \end{array}\left[\begin{array}{ccc:cc} g_3+G & & & & 1 \\ & g_2 & & -1-F & \\ \hdashline -1 & E & 1 & & \\ & -1 & & & H \\ & & -1 & & \end{array}\right]\left[\begin{array}{c} e_{1,2} \\ e_{3,4} \\ e_5 \\ \hdashline j_7 \\ j_8 \end{array}\right]=\left[\begin{array}{c} \\ \\ \hdashline \\ \\ -V \end{array}\right] \tag{6}$$

IV. SENSITIVITY ANALYSIS

All the variables required by sensitivity analysis by the "adjoint network" method [2] are available. The solution of the adjoint system $\mathbf{G}_n{}^T\hat{\mathbf{e}}=\hat{\mathbf{I}}\mathrm{s}$ results in the computation of the variables corresponding to the equation labels in the stamps in tables 1 and 3. The required variables are normal voltages and adjoint currents in voltage sources, and for the controlled sources the voltages and currents in controlling branches in the normal and adjoint networks. Note that the sensitivities relative to the gains of the controlled sources are always equivalent to the sensitivities relative to the corresponding transconductances in the models in fig. 4. Even if the simplified model for the CCVS in fig. 5 is used, the sensitivity can still be obtained as:

$$S_R^{V_o} = -S_{1/R}^{V_o} = -\frac{1}{RV_o}\frac{\partial V_o}{\partial(1/R)} = \frac{1}{RV_o}V_{ab}\hat{V}_{cd} \tag{7}$$

V. CONCLUSION

A compact version of MNA can be obtained by the use of models that use ideal operational amplifiers in the nodal system. The formulation is particularly interesting when the complexities of the use of sparse matrix techniques are to be avoided.

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