STATISTICAL PREDICTION OF ERRORS IN THE LOCALIZATION OF POLES AND ZEROS BY SENSITIVITY ANALYSIS

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ABSTRACT

The paper discusses an accurate method for the prediction of where in the complex plane the poles and zeros of a transfer function of linear circuit will be, taking into account errors in the component values of the circuit. Sensitivity analysis is used, with a generalization of a known process for the determination of pole sensitivity presented. The regions on the complex plane where the poles or zeros are expected to be can be computed as rectangles or, more accurately, as ellipses. Examples with comparisons with Monte Carlo analysis results are presented.

1. INTRODUCTION

In the design of electronic circuits, the careful control of the sensitivity of the design to variations in its component values is of great importance. In the case of filters, the usual methods for the evaluation of the effect of component errors are Monte Carlo analysis, and first-order sensitivity analysis.

The Monte Carlo method usually consists in brute-force simulation of many circuits with the components varying within the expected ranges for the fabrication technology, with the resulting frequency responses plotted for comparison. This method is commonly seen in recent papers. It is reliable, and easily implementable in most circuit simulators, but the analysis can take long time, and the reasons for good or bad results are not always simple to understand.

In principle, the same results can be obtained, for the usual small tollerance components, by first-order sensitivity analysis, at the expense of little more than one analysis of the circuit. Statistical predictions for deviations of gain and phase can easily be obtained. Errors in group delay can also be computed at low cost. There are, however some artifacts resulting from the first-order analysis that must be considered, in particular a tendency to overestimation of the error when the gain changes rapidly with the frequency.

A factor that can complicate the evaluation of the sensitivity of a filter design is that in many applications the absolute gain of a filter is not important, since it can be corrected by automatic gain control circuits, but only the shape of the frequency response curves is significant. With only the analysis of errors in the frequency response it is not always simple to separate the two effects.

An alternative is to look at the errors in the poles and zeros of the filter, that control the shape of the frequency response curves, and are not affected by the absolute gain. Again the two techniques mentioned above can be applied.

This paper discusses the prediction of pole and zero errors by sensitivity analysis. The basic technique is known [1], but we introduce a generalization in the computation of pole sensitivity [2], and two statistical measures for the errors [2]. The measures are compared with Monte Carlo analyses for verification of accuracy and presence of possible artifacts.

While studying this material, we noticed that some standard realizations can present unusually high pole sensitivities for certain approximations. These results are discussed at the end of the paper.

2. TRANSMISSION ZERO SENSITIVITIES

The sensitivity of a complex transmission zero Z_k relative to the variation of a parameter x in the circuit is conveniently defined as:

$$S'^{Z_k}_{x} = x\frac{\partial Z_k}{\partial x}, \tag{1}$$

where the apostrophe denotes that this is a nonnormalized sensitivity measure. A normal sensitivity measure would be normalized (divided by Z_k, in this case). The zero sensitivity can be computed using standard frequency-domain sensitivity analysis as:

$$S'^{Z_k}_{x} = x\frac{\partial Z_k}{\partial x} = \left.\frac{-sS'^{T(s)}_{x}}{S'^{T(s)}_{s}}\right|_{s=Z_k}, \tag{2}$$

where $T(s)$ is the transfer function in Laplace transform that has the zero. The sensitivity of $T(s)$ in relation to s can be obtained by the addition of the sensitivities of $T(s)$

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relative to all the reactive element values (capacitors C_i and inductors L_i), as:

$$S_s^{T(s)} = \sum_{x_i=C_i,L_i} S_{x_i}^{T(s)} = \sum_{x_i=C_i,L_i} x_i \frac{\partial T(s)}{\partial x_i}. \quad (3)$$

All that has to be computed are the partial derivatives of $T(s)$ relative to all the component values at the transmission zero frequency. This can be done conveniently by the use of the adjoint network method [1].

3. POLE SENSITIVITIES

Pole, or natural frequency, sensitivities can be computed by the same process, if the original circuit is modified in a way that transforms poles into transmission zeros. This can be done by zeroing all the inputs of the original network and inserting a current source in series with one branch of the circuit. The transfer function $T(s)$ to be considered is the input impedance seen by this source. The natural frequencies of the original network appear as zeros of this impedance.

Note that the current source can be inserted in series with any branch [2], and not only at the input, as done in [1]. This allows the computation of sensitivities of poles that are nonobservable from the circuit input, as happens with poles of filters built as biquad cascades, or of nonobservable and noncontrollable natural frequencies that appear in some filter structures (as *LC* ladder band-reject filters). Note also that the element in the branch where the current source is inserted must have influence in the pole location, otherwise the impedance zero appears canceled by a pole, and the computation is impossible.

4. MULTIPARAMETRIC MEASURES OF POLE AND ZERO SENSITIVITY

Given the relative variabilities $V_i = \Delta x_i / x_i$ of the circuit component's values, the first-order deterministic error in a zero or pole s_k can be computed as:

$$V_d(s_k) = \sum_{i=1}^{n} V_i S'^{s_k}_{x_i}. \quad (4)$$

This error can be added to the pole or zero complex frequency for the approximate computation of its new position.

When the x_i elements vary in a random way, the pole or zero (in first-order approximation) moves inside a polygon centered at its nominal position, that has as corners some of the points obtained by adding the error above to the original frequency, with the V_i set to plus or minus the maximum values. In practice, it is very improbable that the extreme values of this polygon are reached, because this would require that all the x_i present the maximum error simultaneously.

A more realistic measure is to compute statistical deviations for the pole or zero positions. The simplest measures are the statistical deviations of real and imaginary parts, defined as:

$$V_e(\mathrm{Re}(s_k)) = \sqrt{\sum_{i=1}^{n} \left(V_i \,\mathrm{Re}\left(S'^{s_k}_{x_i}\right)\right)^2}$$
$$V_e(\mathrm{Im}(s_k)) = \sqrt{\sum_{i=1}^{n} \left(V_i \,\mathrm{Im}\left(S'^{s_k}_{x_i}\right)\right)^2} \quad (5)$$

These measures keep the unit used for the V_i: If the V_i are given as variances, the V_e are also variances. If the V_i are number of standard deviations (3σ, usually), The V_e represent also the same number of standard deviations.

They define a rectangle in the complex plane where s_k is expected to be. Tests against Monte Carlo analysis show that the measure is reliable, and it is theoretically exact if the x_i vary following Gaussian distributions.

A problem with this measure is that the correlation between the errors in the real and imaginary parts of s_k due to a single element is being ignored. The actual area where s_k is expected is smaller that the predicted rectangle (although touching its borders), has curved borders, and usually there is more tendency to movement in a certain angle. These facts are easily verified by Monte Carlo analysis.

A better approximation is to consider the region where the s_k poles or zeros are expected as ellipses in the complex plane. The angle of maximum expected variation, that is the angle of the ellipse longer radius with the real axis, can be obtained as:

$$\theta = \angle \sum_{i=1}^{n} \pm \left| V_i S'^{s_k}_{x_i} \right|^2 e^{j\angle S'^{s_k}_{x_i}}, \quad (6)$$

where the plus or minus signs are considered in the way that maximizes the modulus of the summation (it is enough to reduce the error angles to two adjacent quadrants). It is the angle of the longest vector that can be obtained by adding the error vectors with their amplitudes squared.

The two radii of the ellipse are obtained by projecting the error vectors over the corresponding directions, considering Gaussian distributions (the dots mean scalar products):

$$r_{long} = \sqrt{\sum_{i=1}^{n} \left[\left(V_i S'^{s_k}_{x_i}\right) \bullet \left(e^{j\theta}\right)\right]^2}$$
$$r_{short} = \sqrt{\sum_{i=1}^{n} \left[\left(V_i S'^{s_k}_{x_i}\right) \bullet \left(e^{j(\theta - \pi/2)}\right)\right]^2} \quad (7)$$

The result is an ellipse that is approximately inscribed in the original rectangle. Note that if the error vectors are all along two orthogonal directions, it is not possible to determine the angle of maximum expected variation. This, however, rarely occurs.

5. EXAMPLES

The sensitivity measures described above are first exemplified in the filter shown in fig. 1. It is a 4th-order elliptic *LC* singly-terminated filter, coupled to two constant-resistance second-order all pass phase equalizers based in gyrators. This circuit can be used as prototype for active realizations (a doubly terminated version would be less sensitive, but the 4th-order elliptic approximation does not admit an *LC* doubly terminated realization). This is an example where it is necessary to apply the input current source that transforms the natural frequencies into zeros to a branch different from the input. The poles of the equalizers are not observable from the input. The current source was applied in series with the output termination resistor, that affects all the 8 poles.

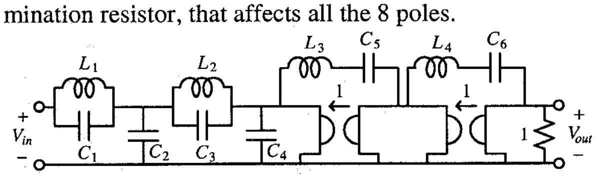

Fig. 1. Filter for the example.
L_1=0.8718; L_2=1.3530; L_3=2.37; L_4=2.17; C_1=0.4428; C_2=1.7018; C_3=0.0595; C_4=1.0069; C_5=0.778; C_6=4.55.

The figures 3-5 show the error predictions for some of the filter's poles and zeros, compared with results of a Monte Carlo analysis with 250 samples. All the circuit elements were assumed as having variabilities of 0.01, or 1% (3 standard deviations), following Gaussian distributions. The filter's poles and zeros (fig. 2) are:

Poles:	Zeros:
–0.211±0.706j (#2)	±3.525j
–0.105±0.994j	±1.610j
–0.230±0.220j	+0.230±0.220j
–0.364±0.479j (#3)	+0.211±0.706j (#1)

The second example illustrates that some designs where low sensitivity is expected can present very high pole sensitivities, and that there are variant designs with much better characteristics in this aspect.

Fig. 6 shows a classical *LC* doubly terminated design for a 5th order Bessel filter. This is one of the two possible designs for this filter with equal terminations, obtained by the classical design method [1], by expansion of the Z_{11} impedance with the characteristic function having all its complex zeros in the left half-plane. The other design would be obtained by distributing the zeros of the characteristic function in both sides of the imaginary axis (this other design is somewhat better in terms of pole sensitivity that the one used as example).

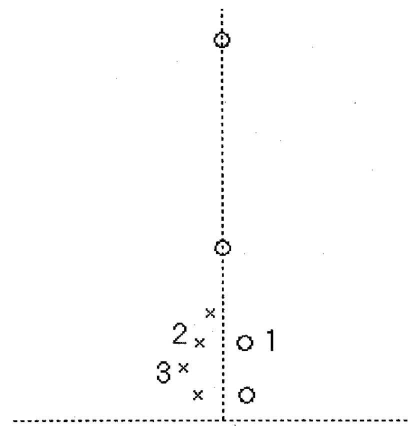

Fig. 2. Poles and zeros of the example filter in fig. 1 (with positive imaginary parts). The indicated zero and poles are used as examples.

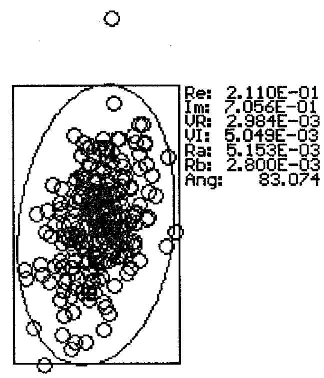

Fig. 3. Zero #1 in fig. 2, with predicted rectangular and ellipsoidal error areas, and results of a Monte Carlo analysis. The zero location and the measures (5), (7), and (6) are listed.

Fig. 4. Pole #2 in fig. 2. Scale reduced by 4 relative to fig. 3.

Fig. 7 shows an alternative design, where the symmetry of the structure is forced by the introduction of additional resistors [2][3]. This design is not practical for a passive realization, due to the attenuation caused by the added resistors, but can be used as prototype for active and digital simulations, where the equalization of dynamic range removes this problem, due to the good sensitivity characteristics.

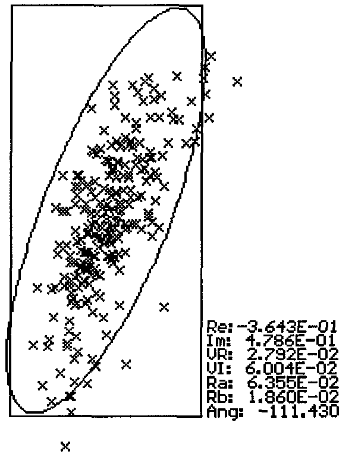

Fig. 5. Pole #3 in fig. 2. Scale reduced by 8 relative to fig. 3.

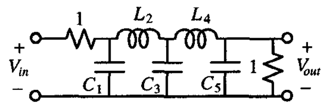

Fig. 6. *LC* doubly terminated 5th-order Bessel filter.
C_1=0.9303; L_2=0.4577; C_3=0.3312; L_4=0.2090; C_5=0.0718.

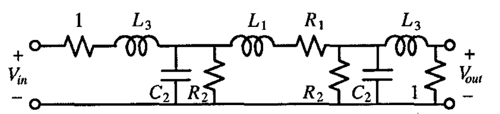

Fig. 7. *RLC* symmetrical 5th-order Bessel filter.
L_1=0.1791; R_1=0.2852; C_2=1.0985; R_2=0.3753; L_3=0.2337.

Fig. 8 shows the previsions for the pole positions obtained by the proposed criterions for both structures, for 1% 3σ random variations in all the elements. The *LC* doubly terminated structure presents extremely high pole sensitivities, orders of magnitude higher than the pole sensitivities of the symmetrical design.

Fig. 9 shows the error prevision for the pole with higher sensitivity in the design in fig. 6, compared with the results of a Monte Carlo simulation with 250 samples (1% variabilities (3σ) with Gaussian distributions for all elements). As expected, the error estimation is not precise when the predicted error is so large. For the other poles, and for the other design, the predictions are accurate.

6. CONCLUSION

Two statistical measures of pole and zero sensitivities were presented, and also a generalization of a method for the computation of pole sensitivities. The statistical predictions of pole and zero positions are accurate for small errors, as exemplified by the low sensitivity zero in fig. 3, and usually still acceptable for larger errors, as in figs. 5.. For very large errors, the ellipsoidal prediction is not so accurate, as seen in fig. 9. It was also shown that even *LC* doubly terminated filters designed for maximum power transfer can exhibit very large pole sensitivities. Of the most usual approximations, this problem appears to be more apparent with the Bessel approximation. The reason appears to be related to the asymmetry of the filter structure, that turns it suboptimal in terms of sensitivity [2][3]. As shown, a symmetrical structure in these cases results in much more precise pole determination.

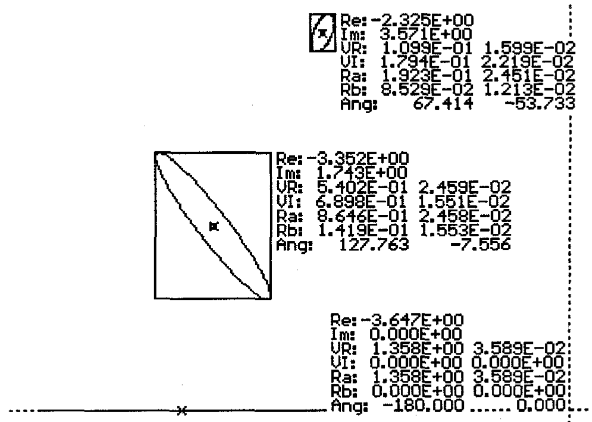

Fig. 8. Poles (upper complex plane) and predicted errors for the filters in figs. 6 and 7. The first errors listed correspond to the filter in fig. 6 and the second to the filter in fig. 7. The errors for the symmetrical design are too small to be seen at this scale.

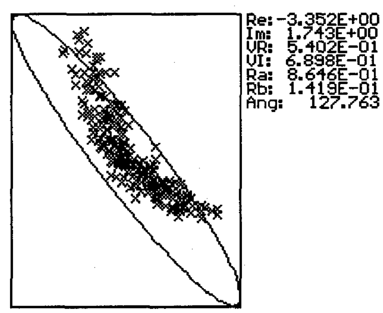

Fig. 9. Error prediction for the most sensitive complex pole of the filter in fig. 6, compared with the actual pole distribution obtained by a Monte Carlo simulation.

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